U419. Let $p > 1$ be a natural number. Prove that

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{\sqrt[p]{k}}-\frac{p}{p-1}\left(n^{\frac{p-1}{p}}-1\right)\right)\in(0,1)$$

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Let $a_n := \sum_{k=1}^{n}\frac{1}{\sqrt[p]{k}}-\frac{p}{p-1}\left(\left(n+1^{\frac{p-1}{p}}\right)-1\right)$ and $b_n := \sum_{k=1}^{n}\frac{1}{\sqrt[p]{k}}-\frac{p}{p-1}\left(n^{\frac{p-1}{p}}-1\right)$.

First we will prove that $a_n \uparrow \mathbb{N}$ and $b_n \downarrow \mathbb{N}$..

Indeed, by Mean Value Theorem there is $c_n \in (n+1, n+2)$ such that

$$a_{n+1}-a_n=\frac{1}{\sqrt[p]{n+1}}-\frac{p}{p-1}\left((n+2)^{\frac{p-1}{p}}-(n+1)^{\frac{p-1}{p}}\right)=$$

$$\frac{1}{\sqrt[p]{n+1}}-\frac{1}{\sqrt[p]{c_n}}>\frac{1}{\sqrt[p]{n+1}}-\frac{1}{\sqrt[p]{n+1}}=0$$

and similarly, there is $c_n \in (n, n+1)$

$$b_n-b_{n+1}=-\frac{1}{\sqrt[p]{n+1}}+\frac{p}{p-1}\left((n+1)^{\frac{p-1}{p}}-n^{\frac{p-1}{p}}\right)=$$

$$\frac{1}{\sqrt[p]{c_n}}-\frac{1}{\sqrt[p]{n+1}}>\frac{1}{\sqrt[p]{n+1}}-\frac{1}{\sqrt[p]{n+1}}=0$$

Since, $a_n \uparrow \mathbb{N}, b_n \downarrow \mathbb{N}$ and $a_n < b_n$, $n \in \mathbb{N}$ then $a_n < b_m$ for any $n, m \in \mathbb{N}$.

Indeed,$a_n < a_{n+m} < b_{n+m} < b_m$.In particular $a_n < b_1$ and $a_1 < b_n$, $n \in \mathbb{N}$.

Therefore, both sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are convergent.

Let $l := \lim_{n\to\infty} a_n$ and $u := \lim_{n\to\infty} b_n$. Since $\lim_{n\to\infty}(b_n - a_n) = 0$ then $l = u$.

Let $c := \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$ and also $a_n < c < b_n$ because $a_n \uparrow \mathbb{N}, b_n \downarrow \mathbb{N}$ and $a_n < b_n$, $n \in \mathbb{N}$.

Thus, we have $a_n < c < b_n \iff$

$$\sum_{k=1}^{n}\frac{1}{\sqrt[p]{k}}-\frac{p}{p-1}\left(\left(n+1^{\frac{p-1}{p}}\right)-1\right)<c<\sum_{k=1}^{n}\frac{1}{\sqrt[p]{k}}-\frac{p}{p-1}\left(n^{\frac{p-1}{p}}-1\right)\iff$$

$$\frac{p}{p-1}\left(n^{\frac{p-1}{p}}-1\right)+c<\sum_{k=1}^{n}\frac{1}{\sqrt[p]{k}}<\frac{p}{p-1}\left(\left(n+1^{\frac{p-1}{p}}\right)-1\right)+c.$$

Since $\frac{p}{p-1}\left(2^{\frac{p-1}{p}}-1\right)\uparrow p \in \mathbb{N}\setminus\{1\}$ then

$$a_1=\sum_{k=1}^{1}\frac{1}{\sqrt[p]{k}}-\frac{p}{p-1}\left(\left((1+1)^{\frac{p-1}{p}}\right)-1\right)=$$

$$1-\frac{p}{p-1}\left(2^{\frac{p-1}{p}}-1\right)\geq 1-\frac{2}{2-1}\left(2^{\frac{2-1}{2}}-1\right)=1-2\left(\sqrt{2}-1\right)=1-\frac{2}{\sqrt{2}+1}=\frac{\sqrt{2}-1}{\sqrt{2}+1}>0$$

and, therefore, $a_1 \leq a_n < c \implies c > 0$.

Also, since $b_1 = \sum_{k=1}^{1}\frac{1}{\sqrt[p]{k}}-\frac{p}{p-1}\left(1^{\frac{p-1}{p}}-1\right)=1$ and $c < b_n \leq b_1$ then $c < 1$.

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